SELF-SIMILAR SOLUTIONS OF ONE-DIMENSIONAL GAS-MOTION PROBLEMS IN A POROUS MEDIUM WITH A TWO-TERM DRAG LAW

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Solutions are obtained in self-similar variables for the equations of gas motion in a porous medium with a two-term drag law for polytropic and adiabatic gas filtration.

An investigation of the self-similar solutions of one-dimensional gas-filtration problems was performed in [1-7]. An analytic solution is obtained in [1] for the problem of plane isothermal gas flow with a linear dependence of the pressure gradient on the velocity. Gas filtration equations with linear and quadratic drag laws and their asymptotic solutions are represented in self-similar variables in [5, 6]. The numerical solution of the self-similar problem of plane isothermal gas filtration with a two-term Darcy law is represented in [7].

Let us consider one-dimensional gas motion in a homogeneous porous medium for a power-law dependence of the gas mass flow rate on the time at the origin.

Polytropic gas filtration with a two-term drag law is described by the following system of equations:

$$\varepsilon \frac{\partial}{\partial t} \rho + \frac{1}{x^{\nu}} \frac{\partial}{\partial x} (x^{\nu} u \rho) = 0, \tag{1}$$

$$-\frac{\partial P}{\partial x} = \frac{u\mu}{k} \left(1 + \frac{\lambda u \rho}{\mu}\right), \tag{2}$$

$$P = Q\rho^{n}. \tag{3}$$

We write the initial and boundary conditions for the system (1)-(3) in the form

$$\rho(x,\ t = 0), \quad \lim_{x \to 0} (x^{\nu} u \rho) = A t^{l}. \tag{4}$$

For $l = \nu(1 + n)(n + 2)^{-1}$ the system (1)-(3) with the conditions (4) describes self-similar motions.

The gas filtration density and velocity in a porous medium are represented in the dimensionless variables:

$$\rho(x,\ t) = \left[\frac{\lambda A^{\frac{3}{1+\nu}}}{Qn\mu} \left(\frac{\varepsilon}{n+2}\right)^{\frac{2\nu-1}{1+\nu}}\right]^{\frac{1+\nu}{(n-1)(1+\nu)+3}} t^{\frac{1}{n+2}} f(\theta), \tag{5}$$

$$u(x,\ t) = \left[\frac{Qn\mu A^{n-1}}{\lambda} \left(\frac{\varepsilon}{n+2}\right)^{1+\nu(n-1)}\right]^{\frac{1}{(n-1)(1+\nu)+3}} t^{-\frac{1}{n+2}} \varphi(\theta), \tag{6}$$

$$\theta = \left[\left(\frac{\varepsilon}{n+2}\right)^{n+1} \frac{\lambda A^{1-n}}{Qn\mu}\right]^{\frac{1+\nu}{(n-1)(1+\nu)+5}} t^{-\frac{n+1}{n+2}} x. \tag{7}$$

In self-similar variables, (1)-(3) and conditions (4) have the form

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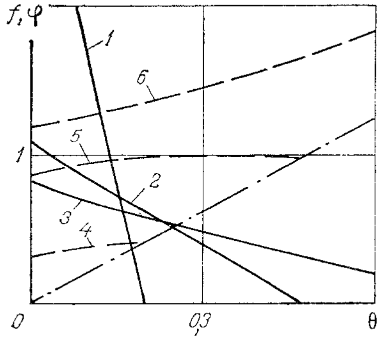

Fig. 1. Gas pressure $f(\theta)$ (solid curves) and gas filtration rate $\varphi(\theta)$ (dashes) in plane isothermal flow for $\sigma = 0$; 1; 50 (curves 1-3, $f(\theta)$; 4-6, $\varphi(\theta)$). The dash-dot curve denotes the asymptotic curve $\varphi = 2\theta$ for the velocity of gas motion for a quadratic drag law ($\sigma = 0$).

$$\frac{1}{\theta^{\nu}}\frac{d(\theta^{\nu}\varphi f)}{d\theta} + f - (n+1)\theta\frac{df}{d\theta} = 0, \tag{8}$$

$$f^{n-1}\frac{df}{d\theta} + \sigma\varphi + f\varphi^2 = 0, \tag{9}$$

$$f(\theta\to\infty) = 0, \ \lim_{\theta\to 0}(\theta^{\nu}\varphi f) = 1. \tag{10}$$

The σ in (9) is determined from the formula

$$\sigma = \frac{\mu}{kQn}\left(\frac{n+2}{\varepsilon}\right)^{\frac{\nu(1+n)}{(n-1)(1+\nu)+3}}\left(\frac{\lambda}{kQn}\right)^{-\frac{2+n(1+\nu)}{(n-1)(1+\nu)+3}}A^{-\frac{n+2}{(n-1)(1+\nu)+3}}$$

We obtain from (9) that $df/d\theta < 0$; consequently, $d(\theta^{\nu}\varphi f)/d\theta < 0$ follows from (8) and the product of the functions $f(\theta)\varphi(\theta)$ decreases to zero as θ grows. Therefore, a coordinate θ^* is found such that $f(\theta)\varphi(\theta) \ll \sigma$ for $\theta > \theta^*$. Here (9) goes over into the linear Darcy law:

$$f^{n-1}\frac{df}{d\theta} + \sigma\varphi = 0. \tag{11}$$

It is shown in [2] that the gas filtration that is described by (8), (11), and the conditions (10) occurs at a finite rate for $n \geqslant 0$.

Let θ_0 be the self-similar coordinate of the gas front, then

$$\varphi(\theta_0) = (n+1)\theta_0. \tag{12}$$

Integrating (11), we obtain that the gas pressure near the front is described by the expression

$$f(\theta) \approx [\sigma n(n+1)\theta_0(\theta-\theta_0)]^{1/n}. \tag{13}$$

Let us examine the numerical solution of the plane isothermal gas-filtration problem in a porous medium $\nu = 0$, $h = 1$.

The system (8), (9) with conditions (10) was integrated numerically by using the Runge–Kutta method for the fourth-order approximation [8]. The numerical solution of the problem was by the method of "adjustment." For $\theta = 0$ an arbitrary value of the velocity $\varphi(\theta=0)=\varphi_0 > 0$ was given, and $f(\theta = 0) = f_0$ was found from (10). Moreover, (8) and (9) are integrated until $\varphi = 2\theta_0$ is satisfied in conformity with (12). If $f(\theta_0) \neq 0$ here, then the value of φ_0 is increased and the computation is repeated. For an abrupt rise in $\varphi(\theta)$, in substance $\varphi(\theta)\to\infty$, the computation ceases, the initial value of φ_0 is diminished and the integration is repeated. Therefore, each iteration narrows the interval to determine $\varphi(\theta=0)$ by half, which permits obtaining numerical solutions for $f(\theta)$ and $\varphi(\theta)$ to the necessary accuracy.

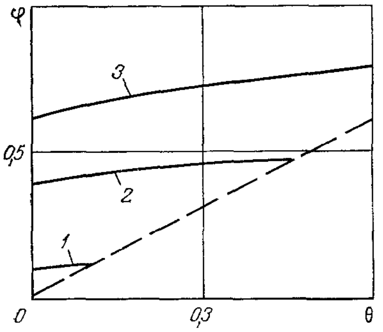

Fig. 2

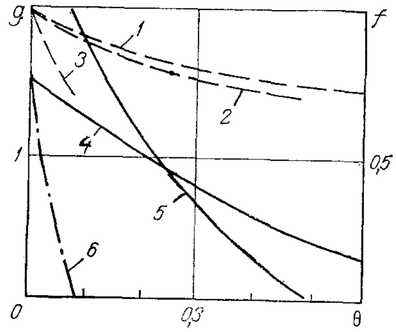

Fig. 3

Fig. 2. Gas filtration rate in an adiabatic flow for $\sigma = 0.02$; 1; 50 (curves 1-3). The dashed curve portrays the function $\varphi = 2\theta$.

Fig. 3. The gas pressure $f(\theta)$ (solid curves) and density $g(\theta)$ (dashed and dash-dot curves) in adiabatic filtration for $\sigma = 0.02$; 1; 50 (curves 1-3 are $f(\theta)$ and curves 4-6 are $g(\theta)$).

Results of a numerical solution of the problem (8), (9) are shown in Fig. 1 for $\nu = 0$ and $n = 1$.

From the formula for σ, we obtain $\sigma = \mu(\lambda A)^{-1}$ for $\nu = 0$ and $n = 1$. The parameter λ is related to the characteristic pore dimension δ and the porosity ε by the expression [9]: $\lambda = 0.012\delta(1-\varepsilon)^{-1}$, consequently $\sigma = 8.5\mu(1-\varepsilon)(A\delta)^{-1}$. For instance, for gas filtration through sand ($\delta = 10^{-4}$ m, $\mu = 0.25\cdot 10^{-4}$ N·sec/m^2, $\varepsilon = 0.1$) we obtain $\sigma \simeq 20A^{-1}$. Consequently, for $A \ll 20$ kg/m^2·sec the gas filtration is described by a linear Darcy law, and for $A \gg 20$ kg/m^2·sec by two-term and quadratic drag laws.

Let us consider self-similar adiabatic gas motions for a two-term drag law, which are described by the system (1)-(2), the energy equation, and the equation of state:

$$\varepsilon\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u H)}{\partial x} = 0, \tag{14}$$

$$E = c_v T,\ H = c_p T,\ P = R\rho T. \tag{15}$$

We used boundary and initial conditions in the form

$$P(x=0,\ t) = At^q,\ \lim_{x\to 0}(u\rho) = B, \tag{16}$$

$$P(x,\ t=0) = P_0,\ \rho(x,\ t=0) = 0. \tag{17}$$

for plane gas motion in a semiinfinite porous medium.

The problem (1)-(2), (14)-(17) is also self-similar with the variable $\theta = x(\lambda B)^{1/2}\cdot(2Ak)^{-1/2}t^{-\frac{q+1}{2}}$. The pressure, density, and velocity for gas filtration are expressed in terms of dimensionless functions of θ:

$$P(x,\ t) = At^q f(\theta),$$
$$\rho(x,\ t) = B\sqrt{\frac{2\lambda B}{Ak}}\, t^{-\frac{q-1}{2}} g(\theta), \tag{18}$$
$$u(x,\ t) = \sqrt{\frac{Ak}{2\lambda B}}\, t^{\frac{q-1}{2}} \varphi(\theta).$$

Here $f(\theta)$, $g(\theta)$, and $\varphi(\theta)$ satisfy equations and conditions corresponding to the problem (1)-(2), (14)-(17):

$$g'[\varphi - (1+q)\theta] + g[\varphi' + 1 - q] = 0, \tag{19}$$

$$f' + \sigma\varphi + g\varphi^2 = 0, \tag{20}$$

$$\varphi'\gamma f + f'[\gamma\varphi - \theta - q\theta] + 2qf = 0, \tag{21}$$

$$f(\theta = 0) = 1;\ \lim_{\theta\to 0}(g\varphi) = 1, \tag{22}$$

$$g(\theta \to \infty) = 0,\ f(\theta \to \infty) = N = \frac{P_0}{A}.$$

Let us analyze the behavior of the functions $f(\theta)$, $g(\theta)$, and $\varphi(\theta)$ from the system (19)-(21) for $q = 0$ and conditions (22). Since $f' < 0$, then it is seen from (21) that for $\gamma\varphi - \theta > 0$ the filtration rate increases as θ grows: $\varphi' > 0$. Here $g' < 0$ and $(\varphi g)' < 0$, while the product $\varphi(\theta) g(\theta)$ diminishes.

The velocity of gas motion $\varphi(\theta)$ has a maximum value at the front equal to $\varphi(\theta = \theta_0) = \theta_0$ for $\theta = \theta_0$. It follows from (19) that for $\varphi = \theta_0$ $g' \to -\infty$. The gas pressure in a porous medium $f(\theta)$ is determined for $\theta < \theta_0$ near the front from the formula

$$f(\theta) = N + \sigma\theta_0(\theta_0 - \theta). \tag{23}$$

The gas filtration velocity $\varphi(\theta)$ and density $g(\theta)$ are expressed here in the form

$$\varphi(\theta) = \theta_0 - \frac{\sigma\theta_0^2(\gamma - 1)}{\gamma N}(\theta_0 - \theta), \tag{24}$$

$$g(\theta) = C_0(\theta_0 - \theta)^{\frac{m}{1+m}}, \tag{25}$$

where C_0 is a constant and $m = [\sigma\theta_0(\gamma - 1)]/\gamma N$.

It is seen from (23) that the gas pressure equals N at the front for $\theta = \theta_0$ while the derivative f' undergoes a jump and gas motion in the porous medium occurs at a finite velocity. For $\theta = \theta_0$ the gas density $g(\theta)$ vanishes, which corresponds to conditions (22).

The following algorithm is used to solve the problem (19)-(22). For $\theta = 0$ values of the gas pressure $f(\theta = 0) = 1$, the gas velocity $\varphi(\theta = 0) = \varphi_0$, and the density $g(\theta = 0) = \varphi_0^{-1}$ are given, where φ_0 is an arbitrary value. Later by using the fourth-order approximation of the Runge–Kutta method [8], the system (19)-(22) is integrated until the value of φ equals θ_0. If φ rises abruptly as θ increases, then the computation ceases, and the value of φ_0 diminishes; then the system of equations is again integrated. For $\varphi(\theta = \theta_0) = \theta_0$ the values $f(\theta = \theta_0)$ and N are compared. If equality holds, then the integral curves for $f(\theta)$, $\varphi(\theta)$ and $g(\theta)$ are determined. In case $f(\theta = \theta_0) < N$ the value of the initial velocity is increased and the computation is repeated. For $f(\theta = \theta_0) > N$ the value of the initial velocity is diminished and integration is again performed.

The dependence of the motion velocity $\varphi(\theta)$ for different σ is shown in Fig. 2 and the pressure $f(\theta)$ for the same σ is displayed in Fig. 3. It is seen that the front coordinate θ_0 equals greater values for large σ.

A self-similar solution of the gas filtration problem is presented in [3] for a linear Darcy law if the gas pressure at the entrance to the porous medium varies exponentially with time.

A self-similar solution also exists for adiabatic gas filtration with a two-term drag law if

$$P(0,\ t) = P_0\exp\{\alpha t\}. \tag{26}$$

The gas pressure, density, and motion velocity are represented in terms of dimensionless variables in the form

$$P(x,\ t) = P_0\exp\{\alpha t\} f(\theta),$$

$$\rho(x,\ t) = B\sqrt{\frac{2B\lambda}{kP_0}}\exp\left\{-\frac{\alpha t}{2}\right\} g(\theta), \tag{27}$$

$$u(x,\ t) = \sqrt{\frac{kP_0}{2\lambda B}}\exp\left\{\frac{\alpha t}{2}\right\}\varphi(\theta),$$

where the self-similar variable is $\theta = x\varepsilon\alpha\sqrt{B\lambda/2kP_0}\exp\{-\alpha t/2\}$.

The equations in self-similar variables follow from (1), (2), (14)-(15):

$$(\varphi g)' + g + \theta g' = 0, \quad (28)$$

$$f' + \sigma\varphi + g\varphi^2 = 0, \quad (29)$$

$$\gamma(f\varphi)' + 2f - \theta f' = 0. \quad (30)$$

Since $P(0, t) = P_0 \exp\{\alpha t\}$ and $\lim\limits_{x\to 0}(\rho u) = B$, then

$$f(\theta = 1) = 1, \ \lim_{\theta\to 0}(\varphi g) = 1. \quad (31)$$

There follows from the initial condition $P(x, t \to -\infty) = 0$ that

$$f(\theta \to \infty) = 0. \quad (32)$$

The system (28)-(30) with the conditions (31) and (32) is solved exactly the same as the problem (19)-(22).

For x = 0 let the gas pressure vary according to the law

$$P(x = 0, t) = P_0 + P_1 \exp\{\alpha t\}. \quad (33)$$

For $P_0 = \mu/k[(\alpha\lambda^2)/k^2]^{-1/3}$ and $P(x, t \to -\infty) = P_0$ there exists an analytic solution of the system (1)-(3) for n = 1. The gas filtration pressure and velocity are represented in the form

$$P(x, t) = P_0 f(\theta),$$
$$u(x, t) = \left(\frac{\alpha k}{\lambda}\right)^{1/3}\varphi(\theta), \quad (34)$$

where

$$\theta = \exp\left\{\left(\frac{\alpha^2\lambda}{k}\right)^{1/3} x - \alpha t\right\}.$$

The system of equations to determine $f(\theta)$ and $\varphi(\theta)$ has the form

$$(f\varphi)' - f' = 0, \quad (35)$$
$$f' + \sigma\varphi + f\varphi^2 = 0.$$

Integrating (35) with the initial and boundary conditions taken into account, we obtain

$$f(\theta) = \sigma\left[1 + \frac{(\alpha k\lambda^2)^{1/3}P_1}{\mu\theta}\right], \quad (36)$$

$$\varphi(\theta) = \left[1 + \frac{\mu\theta}{(\alpha k\lambda^2)^{1/3}P_1}\right]^{-1}.$$

The solutions presented above for gas filtration problems in a porous medium for a two-term drag law can be utilized to analyze the gas pressure, density, and motion velocity dependences.

NOTATION

ε, porosity of the medium; $\rho(x, t)$, gas density; t, time; x, coordinate; u, gas filtration velocity; ν, dimensionality parameter of the problem; μ, gas viscosity; k, permeability of the medium; λ, Forheimer coefficient; n, polytropic index; Q, constant in the polytropy equation; A, constant in the boundary conditions; l, exponent; f, dimensionless pressure; φ, dimensionless filtration rate; θ, self-similar variable; σ, constant in the equations; θ_0, front self-similar coordinate; φ_0, value of φ for $\theta = 0$; f_0, value of f for $\theta = 0$; E, gas internal energy per unit mass; δ, characteristic dimension of the pores; c_v, gas specific heat for constant volume; c_p gas specific heat for constant pressure; R, gas constant; q, exponent in the boundary conditions; B, constant in the boundary conditions; P_0, initial pressure; g, self-similar gas density; γ, ratio of the gas specific heats; N, constant in the boundary conditions; C_0, constant; α, constant in the boundary conditions; P_1, pressure at the boundary; H, enthalpy.

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ANALYSIS OF ELASTIC WAVE DYNAMICS IN WALLS OF A SPHERICAL EXPLOSION CHAMBER

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The wave motion is investigated numerically, and the magnitude of the elastic stresses is estimated in the walls of a spherical explosion chamber.

The theoretical computational model of gasdynamic and mechanical processes proceeding in a spherical explosion chamber was examined in detail in [1]. The proposed model permitted computation of the wave motion parameters within the chamber and estimation of the fraction of energy transmitted to its walls. A detailed comparison between the numerical results obtained and certain experimental—computational data [2] showed good agreement. The investigation executed in [1] permitted the conclusion that the model assures a more rigorous analysis of the phenomena under consideration as compared with the assumptions often used in the literature about the constancy of the pressure on the chamber walls or the possibility of approximating it by the simplest analytic dependences [3, 4].

The present investigation supplements [1] in the numerical study of the dynamics of elastic waves being generated in chamber walls subjected to periodic pulsed impacting loads for the case of finitely or infinitely thick walls, which is of interest for the solution of a broad class of practical problems [2-5].

Within the framework of the model proposed in [1], wave processes in the walls of a spherical explosion chamber of radius ~ 2 m filled with air of the density $\rho_0 = 1.293 \cdot 10^{-3}$ g/cm^3 at a pressure $P_0 = 1$ atm at whose center is an energy-liberating source of 3 cm radius and $E_0 = 7.106 \cdot 10^9$ J energy are considered in this paper. Aluminum ($\rho_1 = 2.7$ g/cm^3) was selected as chamber wall material.

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